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The particle method for the Vlasov–Poisson system using equally spaced initial data points

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Abstract

Some results are given on the convergence of particle methods for the Vlasov–Poisson system of equations. This system describes the time evolution of a collisionless plasma in the presence of an internally consistent electrostatic field. Particle methods are a means for numerically approximating the solutions. For initial data points that are equally spaced a proof of convergence is given in some cases not contained in previous analysis of this problem. It is shown that for once differentiable initial data the particle method converges and the rate is close to linear in the spatial small parameter. For twice differentiable initial data the rate of convergence can be better than linear, and for three times differentiable initial data the rate can be quadratic in the spatial parameter. The proof is based on some different methods than in previous papers on the subject.

1. Introduction

Some results are given on the convergence of particle methods for the Vlasov–Poisson system of equations. This system describes the time evolution of a collisionless plasma in the presence of an internally consistent electrostatic field. The equations under consideration are

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E(x,t) \cdot \nabla_v f = 0, \tag{1.1}$$

$$f(x,v,0) = f_0(x,v),$$

where $E(x,t) = \nabla_x \Phi$ and Φ the solution to

$$\nabla^2 \Phi = 4\pi \int_v (f(x,v,t) - b(x,v))\,\mathrm{d}v. \tag{1.2}$$

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Here $x=(x_1,x_2,x_3)$, $v=(v_1,v_2,v_3)$.

$$\nabla_x=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right),\quad \nabla_v=\left(\frac{\partial}{\partial v_1},\frac{\partial}{\partial v_2},\frac{\partial}{\partial v_3}\right),\quad \nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}.$$

The field $E(x,t)=(E_1,E_2,E_3)$ is therefore given by

$$E_k(x,t)=\int_{\bar{x},v}(f(\bar{x},v,t)-b(\bar{x},v))G_k(x,\bar{x})\,\mathrm{d}v\,\mathrm{d}\bar{x}, \tag{1.3}$$

where

$$G_k(x,\bar{x})=\frac{x_k-\bar{x}_k}{(r(x,\bar{x}))^3}=\frac{x_k-\bar{x}_k}{r^3} \tag{1.4}$$

and $r=r(x,\bar{x})=((x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2+(x_3-\bar{x}_3)^2)^{1/2}$. The function $f(x,v,t)$ represents the distribution of electrons in the presence of a fixed ion background given by $b(x,v)$.

The particle method is a means for numerically approximating the solution to (1.1) and (1.2), and the convergence of this method has been studied by Victory et al., in [5,4,16,15]. In three of these papers the authors develop a theory of convergence for equally spaced initial data points and generalize to three dimensions proofs initially done by Cottet and Raviart in [2,3]. In the fourth the authors give a proof of convergence of the particle method with asymptotically distributed initial data points. The analysis of particle methods based on asymptotically distributed points originates with the work by Neunzert and Wick, [10–12].

In the proofs by Victory et al., much of the analysis is carried out in L_p norms for $p<\infty$, and convergence in the maximum norm is based on the Sobolev imbedding theorem. In these papers a sufficient differentiabilty of the initial functions for (1.1) and (1.2) is required, at least four derivatives bounded in L_∞. Thus, the proofs do not apply for all differentiable solutions of the Vlasov–Poisson system. The Vlasov–Poisson system has classical solutions for initial data with derivatives of all orders, and one can consider extending the convergence theory for particle methods to those differentiable solutions not covered by the theory in these previous papers. We consider equally spaced initial data points and prove the convergence of the particle method for (1.1) and (1.2) when the initial functions are one, two and three times differentiable. The previous proofs do not apply in this case. For once differentiable initial data, the proof is given for more varied initial particle distributions and generalizes to three dimensions the theorem of [18]. A semi-discrete problem is studied, discretized in space but not in time. It is shown that for twice differentiable initial data the convergence of the particle method can be better than linear and for three times differentiable initial data the convergence can be quadratic in the spatial small parameter. The analysis depends on a mollification parameter in the approximate field which approaches zero at a slower rate than the interparticle distance. Some different methods of proof are used in the present paper from those in previous papers. In particular, the analysis does not rely on the theory of singular integrals in L_p spaces for $p<\infty$. The error estimates are obtained directly in the maximum norm.

The type of error bounds obtained for the three-dimensional Vlasov–Poisson system are consistent with those in previous papers and are expressed in terms of a ratio of the interparticle distance parameter, h, to the mollification parameter, δ. However, these estimates may not completely characterize the convergence of the method. For example, the estimates do not account for a convergence of the method when $\delta = \mathrm{O}(h)$, and there is reason to assume that the particle method may be quadratically convergent in this case.

2. Existence and uniqueness of solutions

We begin with some notation. Let R_k refer to k-dimensional Euclidean space, k will be either three or six. Then the following is defined:

$C^n(R_k)$ — the space of continuous, bounded functions on R_k with n continous, bounded partial derivatives.

$C_0^n(R_k)$ — the set of functions in $C^n(R_k)$ with compact support in R_k.

For functions that also depend on time, t, for $0 \leqslant t \leqslant T$ then $C^n(R_k x[0,T])$ is similarly defined. Given $x = (x_1, x_2, x_3) \in R_3$ then $|x| = |x_1| + |x_2| + |x_3|$.

For $g(x) \in C^n(R_3)$ or $f(x,v) \in C^n(R_6)$ and $r = (r_1, r_2, r_3)$, $s = (s_1, s_2, s_3)$, r_k, s_k nonnegative integers then

$$\frac{\partial^{|r|}}{\partial x^r} g(x) = \frac{\partial^{|r|}}{\partial x_1^{r_1} \ldots \partial x_3^{r_3}} g(x)$$

and

$$\frac{\partial^{|r|+|s|}}{\partial x^r \partial v^s} f(x,v) = \frac{\partial^{|r|+|s|}}{\partial x_1^{r_1} \ldots \partial v_3^{s_3}} f(x,v).$$

For system (1.1) and (1.2) it will be assumed that $f_0(x,v) \in C_0^n(R_6)$ for $n \geqslant 1$ and for simplicity we let $b(x,v) = 0$ in (1.2). The characteristic system for (1.1) and (1.2) is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v, \quad x(0) = \xi, \tag{2.1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = E(x,t), \quad v(0) = \eta \tag{2.2}$$

with $E(x,t)$ given by (1.3). The solution to (2.1) and (2.2) is given as $x(t) = x(\xi,\eta,t)$, $v(t) = v(\xi,\eta,t)$, which are continuously differentiable functions of ξ, η and t. The transformation of R_6 given by

$$(\xi,\eta) \to (x(\xi,\eta,t), v(\xi,\eta,t))$$

is measure preserving and invertible. If the inverse mapping is given as

$$\xi = x_0(x,v,t), \quad \eta = v_0(x,v,t),$$

then the solution to (1.1) and (1.2) can be written as

$$f(x,v,t) = f_0(x_0(x,v,t), v_0(x,v,t)) = f_0(\xi,\eta). \tag{2.3}$$

The existence and uniqueness of the solution to (1.1) and (1.2) is stated as follows:

Theorem 2.1. (a) *Let $f_0(x,v)$ be of class $C_0^n(R_6)$, $n \geqslant 1$. Then for any time T the solution to* (1.1) *and* (1.2) *exists and is unique as an element of $C^n(R_6 x[0,T])$.*

(b) *For t in the interval $[0,T]$ the solution $f(x,v,t)$ has bounds*

$$|f(x,v,t)| \leqslant M, \tag{2.4}$$

$$\left|\frac{\partial^m}{\partial t^m} f(x,v,t)\right|, \left|\frac{\partial^{|r|+|s|}}{\partial x^r \partial v^s} f(x,v,t)\right| \leqslant D, \tag{2.5}$$

$$|r| + |s| \leqslant n, \quad m \leqslant n,$$

$$\operatorname{supp} f(x,v,t) \subset \{(x,v), |x| + |v| \leqslant R'\}. \tag{2.6}$$

Proof. See [13,14,9], and additional Refs. [7,17].

3. The discrete approximation

Let

$$\operatorname{supp} f_0(x,v) \subset \Omega_0 = \{(x,v)/|x_k| \leqslant R_0,\ |v_k| \leqslant R_0,\ k=1,2,3\}, \tag{3.1}$$

where $R_0 \leqslant R'$ as given in (2.6). Phase space is then partitioned as follows: let N_x, N_v be positive integers, $h_x = (2R_0)/N_x$, $h_v = (2R_0)/N_v$, and let i,j be multiindices $i=(i_1,i_2,i_3)$, $j=(j_1,j_2,j_3)$. Mesh points (x_i,v_j) are then specified as $x_i = (x_{1,i_1}, x_{2,i_2}, x_{3,i_3})$, $v_j = (v_{1,j_1}, v_{2,j_2}, v_{3,j_3})$ such that for $k=1,2,3$

$$x_{k,i_k} = -R_0 + (i_k - \tfrac{1}{2})h_x, \quad i_k = 1,2,\ldots,N_x,$$
$$v_{k,j_k} = -R_0 + (j_k - \tfrac{1}{2})h_v, \quad j_k = 1,2,\ldots,N_v.$$

The point (x_i,v_j) is therefore the center of the (i,j)th "rectangle" given by

$$S_{i,j} = \left\{(x,v)/x_{k,i_k} - \frac{h_x}{2} \leqslant x_k \leqslant x_{k,i_k} + \frac{h_x}{2}, v_{k,j_k} - \frac{h_v}{2} \leqslant v_k \leqslant v_{k,j_k} + \frac{h_v}{2},\ k=1,2,3\right\}.$$

We will make use of the notation:

$\Delta x = (h_x)^3$ volume in position space,
$\Delta v = (h_v)^3$ volume in velocity space,
$h = 3(h_x + h_v)$ interparticle distance.

For $\xi = x_i$, $\eta = v_j$ then the solutions to (2.1) and (2.2) are given by $x(t) = x(i,j,t)$, $v(t) = v(i,j,t)$. The objective is to approximate the trajectories $(x(i,j,t), v(i,j,t))$.

A discrete approximation to system (2.1) and (2.2) is defined as follows:

let $(\bar{x}(i,j,t), \bar{v}(i,j,t))$ be solutions to

$$\frac{dx}{dt} = v, \quad x(0) = x_i, \tag{3.2}$$

$$\frac{dv}{dt} = \bar{E}(x,t), \quad v(0) = v_j, \tag{3.3}$$

where $\bar{E}(x,t) = (\bar{E}_1(x,t), \bar{E}_2(x,t), \bar{E}_3(x,t))$ and

$$\bar{E}_k(x,t) = \sum_{l,m} \Delta x \Delta v f_0(x_l, v_m) G_{\delta,k}(x, \bar{x}(l,m,t)). \tag{3.4}$$

Here

$$G_{\delta,k}(x,\bar{x}) = \int_{R_3} G_k(x, \bar{x} - z)\omega_\delta(z)\,\mathrm{d}z. \tag{3.5}$$

The function $\omega_\delta(z)$ is defined by
(1) $\omega(z) = \omega(\|z\|)$, $\|z\| = (\sum_{k=1}^{3} z_k^2)^{1/2}$ (radially symmetric),
(2) $\omega(z) = 0$, $\|z\| \geqslant 1$ (compact support),
(3) $\int_{R_3} \omega(z)\,\mathrm{d}z = 1$.
then

$$\omega_\delta(z) = \frac{1}{\delta^3}\omega(z/\delta),$$

an approximation to the delta function.

Three functions are considered:

(1)

$$\omega_\delta^1(z) = \begin{cases} \dfrac{3}{4\pi\delta^3}, & \|z\| \leqslant \delta, \\ 0, & \|z\| > \delta, \end{cases}$$

$$G_{\delta,k}(x,\bar{x}) = \begin{cases} \dfrac{x_k - \bar{x}_k}{(r(x,\bar{x})^3}, & r(x,\bar{x}) > \delta, \\ \dfrac{x_k - \bar{x}_k}{\delta^3}, & r(x,\bar{x}) \leqslant \delta, \end{cases}$$

$\omega_\delta^1(z) \in L^\infty(R_3)$, $G_{\delta,k}(x,\bar{x})$ is of class $C^0(R_3 x R_3)$.

(2)

$$\omega_\delta^2(z) = \begin{cases} \dfrac{3}{4\pi\delta^3}[1 + 5(\|z\|/\delta)^2 - 6(\|z\|/\delta)^3], & \|z\| \leqslant \delta, \\ 0, & \|z\| > \delta, \end{cases}$$

$$G_{\delta,k}(x,\bar{x}) = \begin{cases} \dfrac{x_k - \bar{x}_k}{(r(x,\bar{x}))^3}, & r(x,\bar{x}) > \delta, \\ \dfrac{x_k - \bar{x}_k}{\delta^3}[1 + 3(r/\delta)^2 - 3(r/\delta)^3], & r(x,\bar{x}) \leqslant \delta, \end{cases}$$

$\omega_\delta^2(z) \in C_0(R_3)$ and positive, $G_{\delta,k}(x,\bar{x}) \in C^1(R_3 x R_3)$.

(3)

$$\omega_\delta^3(z) = \begin{cases} \dfrac{3}{4\pi\delta^3}[21/2 - 567/2(\|z\|/\delta)^4 + 504(\|z\|/\delta)^5 - 231(\|z\|/\delta)^6], & \|z\| \leqslant \delta, \\ 0, & \|z\| > \delta, \end{cases}$$

$$G_{\delta,k}(x,\bar{x}) = \begin{cases} \dfrac{x_k - \bar{x}_k}{(r(x,\bar{x}))^3}, & r(x,\bar{x}) > \delta, \\ \dfrac{x_k - \bar{x}_k}{\delta^3}[\frac{21}{2} - \frac{243}{2}(\frac{r}{\delta})^4 + 189(\frac{r}{\delta})^5 - 77(\frac{r}{\delta})^6], & r(x,\bar{x}) \leqslant \delta, \end{cases}$$

$\omega_\delta^3(z) \in C_0^1(R_3)$. In addition

$$\int_{R_3} \|z\|^2 \omega_\delta^3(z)\,\mathrm{d}z = 0. \tag{3.6}$$

$G_{\delta,k}(x,\bar{x}) \in C^2(R_3 x R_3)$. Condition (3.6) gives the result that the first three moments of $G_k(x,\bar{x}) - G_{\delta,k}(x,\bar{x})$ are zero.

It will be assumed that $\delta = h^u$, $0 < u < 1$. Thus, δ approaches zero at a slower rate than the interparticle distance parameter, h. Let

$$e(t) = \max_{i,j}(|x(i,j,t) - \bar{x}(i,j,t)| + |v(i,j,t) - \bar{v}(i,j,t)|),$$

which is the distance between exact and approximate trajectories. The theorem on the convergence of particle methods is:

Theorem. *In the statements that follow the constants $K_1, K_2, K_3, \varepsilon$ depend on the initial function for (1.1), $f_0(x,v)$, time T, and u.*

(1) *If the solution, $f(x,v,t)$, to (1.1),(1.2) is of class $C^1(R_6 x[0,T])$ with bounded support in R_6 for each t, and if in (3.4) and (3.5) $\omega_\delta(z) = \omega_\delta^1(z)$ with $\delta = h^u$, $0 < u < 1$ then*

$$e(t) \leqslant K_1(\delta^2 + h|\ln(\delta)|)$$

for $h \leqslant \varepsilon$ and $t \in [0,T]$. If $\frac{1}{2} \leqslant u < 1$ then

$$e(t) \leqslant K_1 h|\ln(h)|,$$

close to linear.

(2) *If $f(x,v,t) \in C^2(R_6 x[0,T])$ with bounded support in R_6 and if $\omega_\delta(z) = \omega_\delta^2(z)$ with $\delta = h^u$, $0 < u < 1$ then*

$$e(t) \leqslant K_2 \left(\delta^2 + \frac{h^2}{\delta}\right)$$

for $h \leqslant \varepsilon$ and $t \in [0,T]$. If $\delta = h^{2/3}$ then

$$e(t) \leqslant K_2 h^{4/3},$$

better than linear.

(3) *If $f(x,v,t) \in C^3(R_6 X[0,T])$ with bounded support in R_6 and if $\omega_\delta(z) = \omega_\delta^3(z)$ with $\delta = h^u$, $0 < u < \frac{3}{4}$ then*

$$e(t) \leqslant K_3 \left(\delta^4 + \frac{h^3}{\delta^2}\right)$$

for $h \leqslant \varepsilon$ and $t \in [0,T]$. If $\delta = h^{1/2}$ then

$$e(t) = K_3 h^2,$$

quadratic convergence.

The error bounds obtained in the theorem are consistent with the type of bounds obtained in [4] in three dimensions and in [2] in one dimension. However, the convergence of the particle method may not be entirely described by these estimates. One indication of this is that they do not account for a convergence of the method when $\delta = O(h)$, and even if they did apply in this case they would indicate at best a linear rate of convergence. However the particle method with equally spaced initial points may be quadratically convergent for $\delta = O(h)$. One reason for this assumption is that the related point vortex method has been shown to be a second-order accurate method [1,6,8]. Another is that experimental results on the one-dimensional Vlasov–Poisson system indicate a quadratic convergence in the spatial parameter for $\delta = O(h)$. Thus, for cases 2 and 3 above it may be that the estimates should indicate $O(h^2)$ convergence as $\delta \to h$. We can conjecture that a more precise analysis may show that for

(1)

$$e(t) \leqslant K_1(\delta^2 + h),$$

(2)

$$e(t) \leqslant K_2(\delta^2 + h^2),$$

(3)

$$e(t) \leqslant K_3\left(\delta^4 + \frac{h^3}{\delta}\right).$$

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